Lesson 3

- Discrete transfer function
- Frequency behaviour LTI systems
- Relation between transfer function and impuls response
- Questions / exercises

Transfer function

Each LTI system can be written as (for FIR systems applies: $a_i = 0$):

$$y[n] = \sum_{i=0}^{M} b_i \cdot x[n-i] - \sum_{i=1}^{N} a_i \cdot y[n-i]$$

In the z-domain applies:

$$Y(z) = \sum_{i=0}^{M} b_i \cdot \frac{X(z)}{z^i} - \sum_{i=1}^{N} a_i \cdot \frac{Y(z)}{z^i} = X(z) \cdot \sum_{i=0}^{M} \frac{b_i}{z^i} - Y(z) \cdot \sum_{i=1}^{N} \frac{a_i}{z^i} \quad \rightarrow$$

$$Y(z) \cdot \left(1 + \sum_{i=1}^{N} \frac{a_i}{z^i}\right) = X(z) \cdot \sum_{i=0}^{M} \frac{b_i}{z^i} \quad \rightarrow \quad \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{M} \frac{b_i}{z^i}}{1 + \sum_{i=1}^{N} \frac{a_i}{z^i}}$$

Transfer function

The transfer function H(z) is the ratio between output Y(z) and input X(z):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{M} \frac{b_i}{z^i}}{1 + \sum_{i=1}^{N} \frac{a_i}{z^i}}$$

Example: Given the third order LTI system:

y[n] = 2·x[n] – x[n-2] + 0.5 · y[n-1] - 0.4 · y[n-2] + 0.25 · y[n-3]

The transfer function of this system is then given by:

$$H(z) = \frac{2 - \frac{1}{z^2}}{1 - \frac{0.5}{z} + \frac{0.4}{z^2} - \frac{0.25}{z^3}} = \frac{2 \cdot z^3 - z}{z^3 - 0.5 \cdot z^2 + 0.4 \cdot z - 0.25}$$

Remark: it is common to write the transfer function as a quotient of 2 polynomials (with non-negative integer exponents)

Frequency behaviour LTI systems

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For LTI systems, the transfer function can be used to determine the amplification of signals as a function of the frequency.

Consider a sinelike signal: $x[n] = e^{j\cdot\Omega\cdot n}$

We define y[n] as: $y[n] = x[n-1] = e^{j\cdot\Omega\cdot(n-1)} = e^{-j\cdot\Omega}\cdot e^{j\cdot\Omega\cdot n}$

For the Z-transforms of x[n] and y[n] apply:

$$X(z) = \sum_{n=0}^{\infty} e^{j\cdot\Omega\cdot n}\cdot z^{-n}$$

$$Y(z) = \sum_{n=0}^{\infty} e^{-j\cdot\Omega}\cdot e^{j\cdot\Omega\cdot n}\cdot z^{-n} = e^{-j\cdot\Omega}\cdot\sum_{n=0}^{\infty} e^{j\cdot\Omega\cdot n}\cdot z^{-n} = e^{-j\cdot\Omega}\cdot X(z)$$

Since y[n] = x[n-1], it also applies: $Y(z) = z^{-1}\cdot X(z)$

Combining both equations gives: $z = e^{j\cdot\Omega}$

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Example of frequency behaviour

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Given: y[n] = 0.156·(x[n] - x[n-1] + x[n-2]) + 1.5·y[n-1] - 0.8·y[n-2]

The transfer function of this system is:

$$H(z) = \frac{0.156 \cdot \left(1 - \frac{1}{z} + \frac{1}{z^2}\right)}{1 - \frac{1.5}{z} + \frac{0.8}{z^2}} = \frac{0.156 \cdot \left(z^2 - z + 1\right)}{z^2 - 1.5 \cdot z + 0.8}$$

The amplification of the system as a function of Ω is then:

$$H(\Omega) = \frac{0.156 \cdot \left(e^{j \cdot 2 \cdot \Omega} - e^{j \cdot \Omega} + 1\right)}{e^{j \cdot 2 \cdot \Omega} - 1.5 \cdot e^{j \cdot \Omega} + 0.8}$$

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The graphs of the absolute value and angle (phase shift) of H(Ω) are shown on the next page.

Example of frequency behaviour

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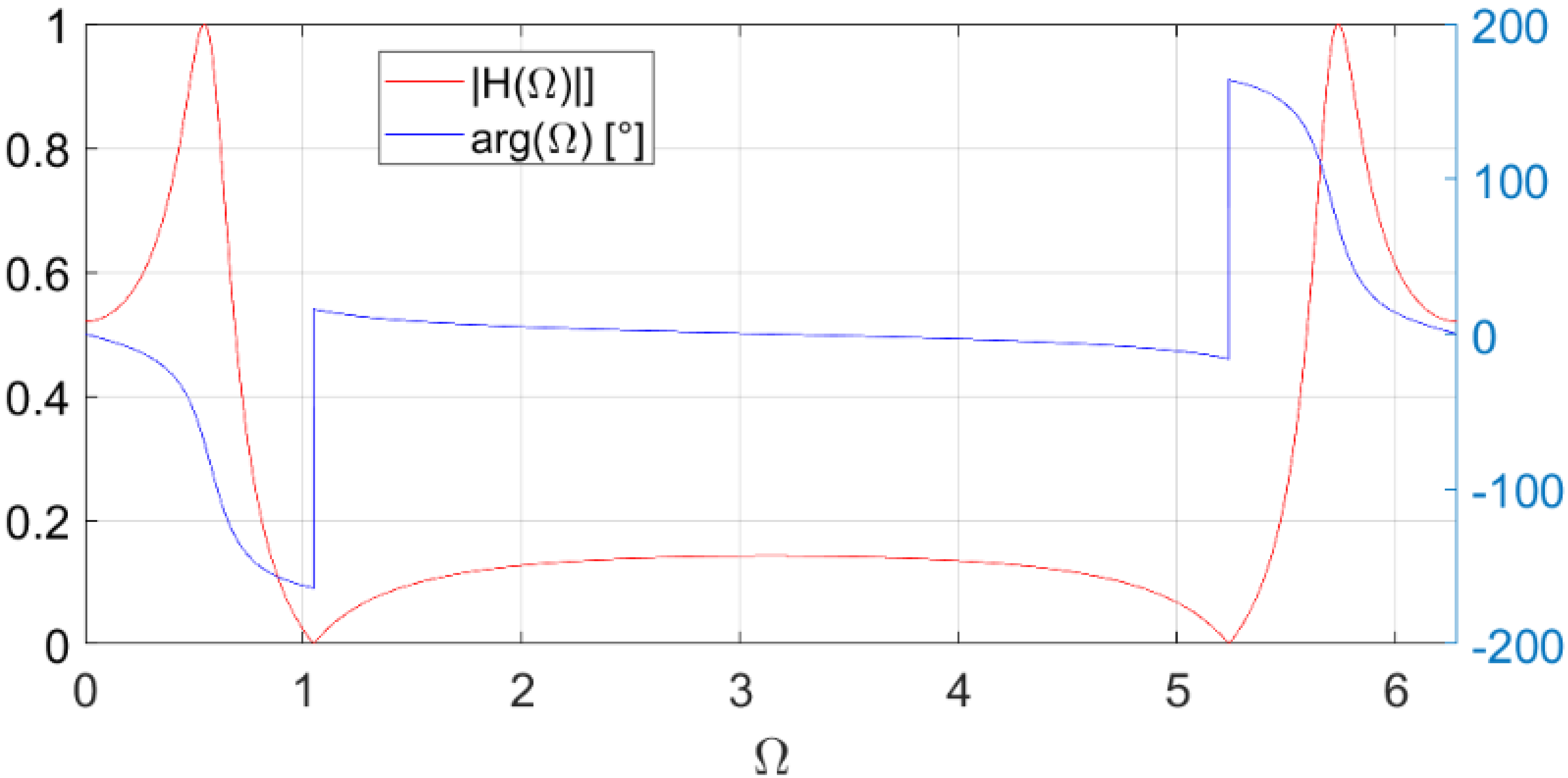

Matlab script (in rough lines):

```
W = 0 : 0.001 : 2*pi;                          % discrete frequency
z = exp(1j*W);
H = 0.156*(z.*z-z+1)./(z.*z-1.5*z+0.8);      % transfer function
plot (W, abs(H), 'red‘); grid on;
figure; plot (W, angle(H), 'blue'); grid on;
```

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Transfer function = impuls response

The following applies:

$$\boldsymbol{H(z) = \frac{Y(z)}{X(z)} \quad \rightarrow \quad Y(z) = H(z) \cdot X(z)}$$

Consider: x[n] = δ[n] → X(z) = 1 → Y(z) = H(z)

So, the transfer function H(z) equals the z-transform of the impuls respons.

Example impulse response

Assume: y[n] = 5· x[n] - 3.6·x[n-1] +1.4· y[n-1] - 0.48·y[n-2]

The first 10 values of the impuls response can be calculated directly from the diffence equation:

{ 5, 3.4, 2.36, 1.672, 1.208, 0.8886, 0.6643, 0.5034, 0.3859, 0.2987 }

The transfer function equals:

$$H(z) = \frac{5 - \frac{3.6}{z}}{1 - \frac{1.4}{z} + \frac{0.48}{z^2}} = \frac{5 \cdot z^2 - 3.6 \cdot z}{z^2 - 1.4 \cdot z + 0.48} = \frac{5 \cdot z^2 - 3.6 \cdot z}{(z - 0.8) \cdot (z - 0.6)} \rightarrow$$

$$H(z) = \frac{2 \cdot z}{z - 0.8} + \frac{3 \cdot z}{z - 0.6}$$

The impuls response then equals: $\mathrm{h}[\mathrm{n}] = 2 \cdot 0.8^{\mathrm{n}} + 3 \cdot 0.6^{\mathrm{n}}$

The first 10 values equal according to this equal: { 5, 3.4, 2.36, 1.672, 1.208, 0.8886, 0.6643, 0.5034, 0.3859, 0.2987 }

Questions / Exercises

1. Give the tranfer functions of the difference equations given below.

a. y[n] = (x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5] + x[n-6]) / 7

b. y[n] = x[n] - x[n-2] + 0.9 · y[n-2]

c. y[n] = 2·x[n] – 1.6·x[n-1] + 0.8·y[n-1]

2. Give the difference equation of the following transfer functions.

a. $H(z) = \frac{1}{z-1}$

b. $H(z) = \frac{2 \cdot z}{1-0.8 \cdot z+0.5 \cdot z^2}$

c. $H(z) = \frac{3}{z+0.5} - \frac{2}{z-0.7}$

d. $H(z) = \frac{z}{2-0.8 \cdot z+0.5 \cdot z^2}$

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3. Calculate the first 10 samples of the impuls response of the systems described by the transfer functions of question 2.

Questions / Exercises

4. Assume an analogue sine x(t) given by: $x(t) = X_{top} \cdot \sin(\omega \cdot t+\varphi_0)$

The signal x(t) is sampled at a sample frequency of f_{sample}.

a. Give the relation between the angular frequency of the discrete signal X[n] and ω and f_{sample}.

b. What is the range of the angular frequency of x[n] if the sampling meets the theorema of Shannon?

5. Reason that the steady state value of a step response equals:

$y[n \to \infty] = H(1)$

b. Calculate the steady state value of the step response of the systems given by question 1 and 2.

Questions / Exercises

6. Given a LTI system with difference equation:

$$y[n] = 0.4 \cdot x[n] + 0.9 \cdot y[n-1]$$

a. Calculate the steady state step response of this system.

b. Assume the input signal of the system is obtained from an analogue signal $x(t) = 2 \cdot \sin(\omega \cdot t)$ that is sampled by a frequency of 10 [kHz]. Calculate the amplitude and phase shift of the output y[n] when f=200[Hz].

c. Give in a graph the frequency response plot of this system (you don't have to use logarithmic scales). Plot it in the range: $0 \leq \Omega \leq \pi$. Use matlab to make the plot.

d. What kind of filter is this LTU system?