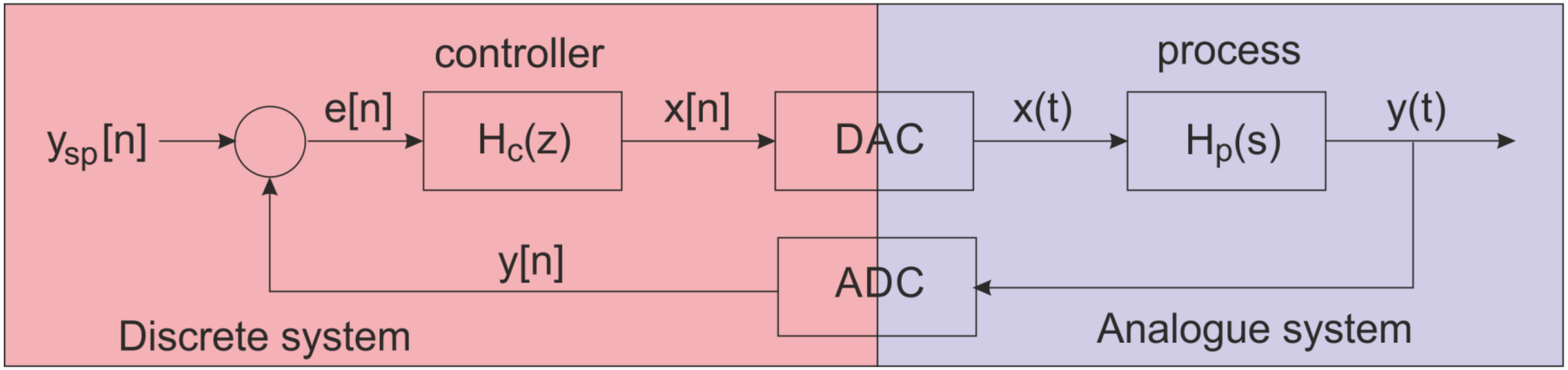
Lesson 5

- Combination of analogue and digital systems
- Relation between s (Laplace) and z (discrete)
- Euler approximation
- Tustin approximation
- Questions / exercises

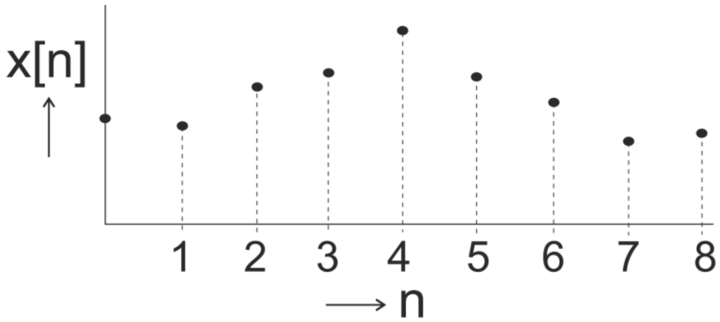
Combination of discrete and analogue systems

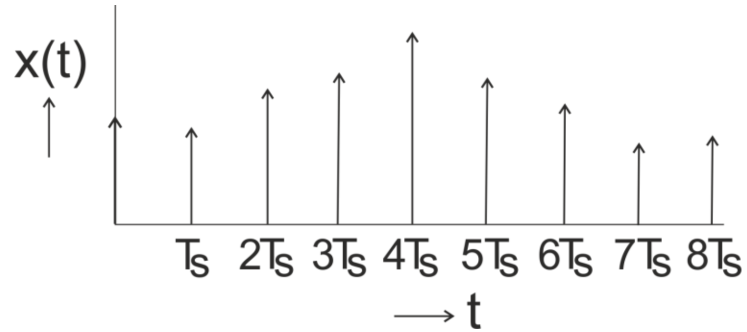


$$Y(z,s) = \frac{H_c(z) \cdot H_p(s)}{1 + H_c(z) \cdot H_p(s)} \cdot Y_{sp}[z] \qquad ???$$

Relation between z and s

Consider a function x(t) that is sampled with a sample time of T_s.





The z-transform of x[n] can be written as: $X(z) = \sum_{n=0}^{\infty} x[n] \cdot z^{-n}$

The discretized signal x(t) can be written as a sum of dirac functions δ(t):

$$x(t) = \sum_{n=0}^{\infty} x(n \cdot T_s) \cdot \delta(t - n \cdot T_s) \;\rightarrow\; X(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x(n \cdot T_s) \cdot \delta(t - n \cdot T_s) \cdot e^{-s \cdot t} \cdot dt \;\rightarrow$$

$$X(s) = \sum_{n=0}^{\infty} x(n \cdot T_s) \cdot e^{-s \cdot n \cdot T_s} = \sum_{n=0}^{\infty} x[n] \cdot e^{-s \cdot T_s \cdot n}$$

X(z) and X(s) equal if: $z = e^{s \cdot T_s} \;\rightarrow\; s = \frac{ln(z)}{T_s} = f_s \cdot ln(z) \quad f_s = \frac{1}{T_s}$

Euler approximation

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Analogue systems can be combined with discrete systems by the substitution of s by: $s \rightarrow f_s \cdot \ln(z)$.

However, this substitution has a big disadvantage:

the numerator and denumerator of the overall transfer function are no polynomial functions anymore. This makes a the system description in terms of poles and zeros, stability, bode diagrams, etc. as you learned it impossible.

Euler gives an approximation of $z = e^{s \cdot T_s}$ in such a way that the overall transfer function remains a quotient of 2 polynomials, so all theorie discussed remains valid.

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Forward Euler (1)

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Assume the function: $s = f_s \cdot ln(z)$

The first order Taylor-series approximation of s around z=1 equals:

$$s \approx s(z=1) + \left.\frac{ds}{dz}\right]_{z=1} \cdot (z-1) = f_s \cdot (z-1)$$

This approximation is called the forward difference Euler approximation (or simply forward Euler).

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Forward Euler (2)

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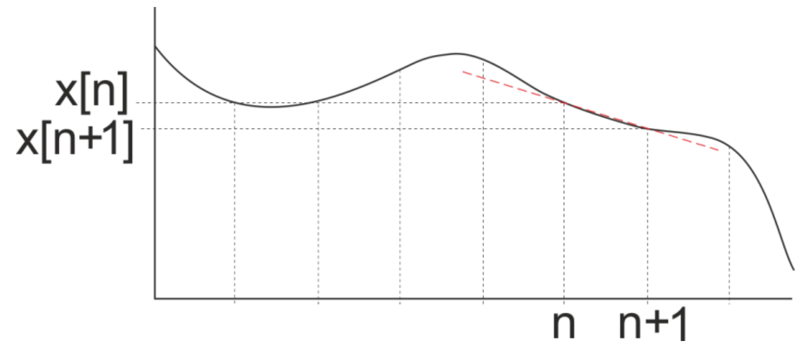
Another way to look at it:

A derivation in the time domain corresponds to a multiplication of s in the s-domain. So:



$$y(t) = \frac{dx}{dt} \quad \rightarrow \quad Y(s) = X(s) \cdot s$$

In the time-domain, we can approximate a derivation by:

$$y(t) = \frac{dx}{dt} \approx \frac{x[n+1] - x[n]}{T_s} \quad \rightarrow \quad Y(z) = \frac{z \cdot X(z) - X(z)}{T_s} = X(z) \cdot f_s \cdot (z-1)$$

We see that a derivation in the time domain corresponds to a multiplication with f_s·(z-1) in the z-domain. This means that the Laplace operator s corresponds to f_s·(z-1) in the z-domain

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Backward Difference Euler

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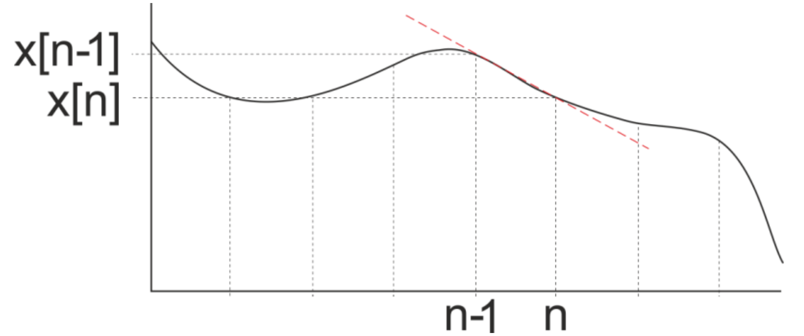
Instead of looking one sample ahead, you can also approximate the derivation of $x(n \cdot T_s)$ by looking one sample back:



$$y(t) = \frac{dx}{dt} = \frac{x[n] - x[n-1]}{T_s} \quad \rightarrow \quad Y(z) = \frac{X(z) - \frac{X(z)}{z}}{T_s} = X(z) \cdot f_s \cdot \left(1 - z^{-1}\right)$$

We see that a derivation in the time domain corresponds to a multiplication with $f_s \cdot (1-z^{-1})$ in the z-domain. This means that the Laplace operator s corresponds to $f_s \cdot (1-z^{-1})$ in the z-domain

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Example Euler

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Assume a second order Butterworth low pass filter, given by:

$$H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + \sqrt{2}\cdot\frac{s}{\omega_0} + 1}$$

The corresponding discrete transfer function H(z) of it is given by:

$$H(z) = \frac{1}{\left(\frac{f_s}{\omega_0}\right)^2 \cdot (1 - z^{-1})^2 + \sqrt{2}\cdot\frac{f_s}{\omega_0}(1 - z^{-1}) + 1} \rightarrow$$

$$H(z) = \frac{z^2}{\left(\left(\frac{f_s}{\omega_0}\right)^2 + \sqrt{2}\cdot\frac{f_s}{\omega_0} + 1\right)\cdot z^2 - \left(2\cdot\left(\frac{f_s}{\omega_0}\right)^2 + \sqrt{2}\cdot\frac{f_s}{\omega_0}\right)\cdot z + \left(\frac{f_s}{\omega_0}\right)^2}$$

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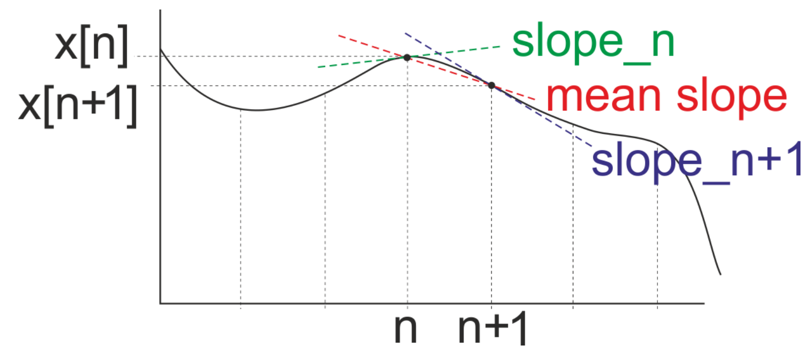
This equation can be used to derive the difference equation.

Tustin

Tustin provides a more accurate approximation of s. It assumes that the quotient $\frac{x[n+1]-x[n]}{T_s}$ is the average of the derivative of x(t) at t=n·T_s and t=(n+1)·T_s.



$$y(t) = \frac{dx}{dt} \quad and: \quad \frac{x[n+1]-x[n]}{T_s} = \frac{y[n+1]+y[n]}{2} \quad \rightarrow$$

$$z \cdot Y(z) + Y(z) = \frac{2}{T_s}\left(z \cdot X(z) - X(z)\right) \;\rightarrow\; Y(z) = X(z) \cdot 2 \cdot f_s \cdot \frac{z-1}{z+1}$$

A derivation in the time domain corresponds to a multiplication with 2·f_s·(z-1)/(z+1) in the z-domain. According to Tustin, the Laplace operator s corresponds to 2·f_s·(z-1)/(z+1) in the z-domain.

Example Tustin

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Assume a first order high pass filter, given by:

$$H(s) = \frac{\frac{s}{\omega_0}}{\frac{s}{\omega_0} + 1}$$

The corresponding discete transfer function H(z) using Tustin is given by:

$$H(z) = \frac{2 \cdot \frac{f_s}{\omega_0} \cdot \frac{z-1}{z+1}}{2 \cdot \frac{f_s}{\omega_0} \cdot \frac{z-1}{z+1} + 1} = \frac{2 \cdot \frac{f_s}{\omega_0} \cdot (z-1)}{\left(2 \cdot \frac{f_s}{\omega_0} + 1\right) \cdot z + 1 - 2 \cdot \frac{f_s}{\omega_0}}$$

This equation can be used to derive the difference equation.

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Questions / Exercises

1. Give the transfer function H(z) and difference equation of the following analogue systems. Use Euler backward:

a. $H(s) = \frac{1}{1+s\cdot\tau}$ $\quad where: \tau = 0.01\ [s]$**. The sample rate is: f_s=1 [kHz]**

b. $H(s) = \frac{1}{1+s\cdot\tau}$ $\quad where: \tau = 0.01\ [s]$**. The sample rate is: f_s=100 [kHz]**

c. $H(s) = \frac{s\cdot 0.1}{(1+s\cdot 0.02)\cdot(1+s\cdot 0.005)}$. **The sample rate is: f_s=100 [kHz]**

2. Given a PID controller: $H(s) = 5 + 2\cdot s + \frac{0.3}{s}$

Give the transfer function H(z) and the difference equation at a sample rate of 500 [samples/s]. Use Tustin.

Questions / Exercises

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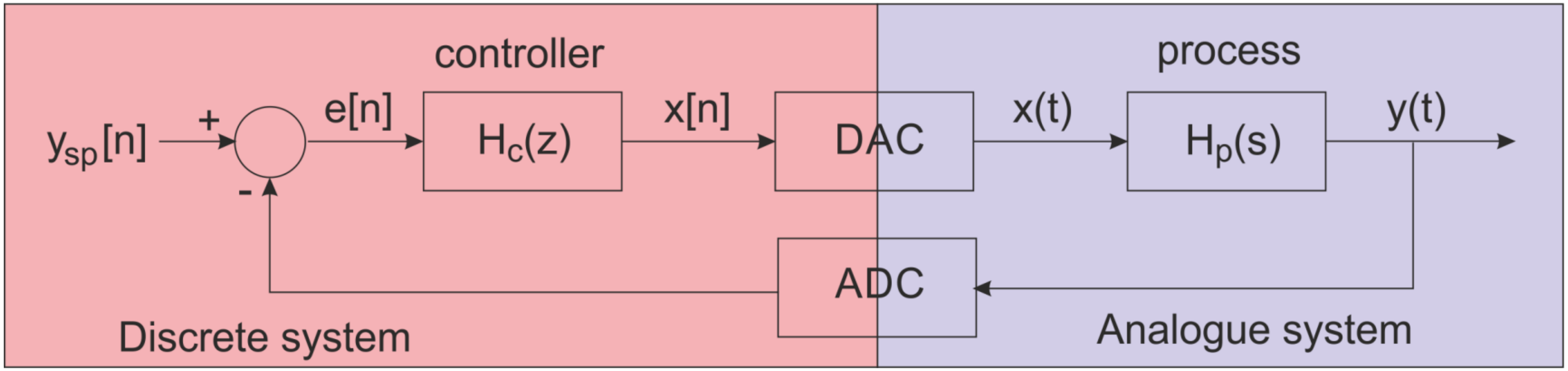
3. A first order (analogue) process is described by: $H_p(s) = \frac{6}{1+s \cdot 0.5}$

The process is controlled via a discrete P-controller (see figure) with a proportional constant of 5. The sample rate is 20 [samples/s].



a. Give the overal transfer function of the system, defined as:

$H(z) = Y(z) / Y_{sp}(z)$. Use Tustin.

b. Calculate the step response y[n] ($y_{sp}[n] = u[n]$).

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Questions / Exercises

4. Consider the system of question 3, only now we control the process via a discrete PI-controller. The proportional constant is 5, the sample rate is 20 [samples/s]. The integral constant is choosen in such a way that the system is critically damped.

a. Descripe when we can speak of a critically damped system.

b. Calculate the gain of the I-controller.

c. Give the difference equation of the discrete PI-controller.

d. Show in a graph the discrete step response y[n].